A WHITEHEAD ALGORITHM FOR TORAL RELATIVELY HYPERBOLIC GROUPS

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Abstract. The Whitehead problem is solved in the class of toral relatively hyperbolic groups G (i.e. torsion-free relatively hyperbolic groups with abelian parabolic subgroups): there is an algorithm which, given two finite tuples $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ of elements of G, decides whether there is an automorphism of G taking u_i to v_i for all i.

1. The result

Consider the following problem about a group G and its automorphisms: given finite tuples $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ of elements of G, decide whether there is an automorphism of G taking u_i to v_i for all i, and, if so, find one. A classical result due to Whitehead solved this problem for finitely generated free groups, see [10] or [9]. We shall refer to it as the *Whitehead Problem* (WhP) for G. There is a natural generalization called the *generalized WhP*: given finite tuples $(u_{11}, \dots, u_{1n_1}, \dots, u_{k1}, \dots, u_{kn_k})$ and $(v_{11}, \dots, v_{1n_1}, \dots, v_{k1}, \dots, v_{kn_k})$ of elements of G, decide whether there is an automorphism of G taking u_{ij} to $v_{ij}^{g_i}$ for all i, j, and some $g_i \in G$.

Apart from the classical Whitehead result, there has been progress in several directions about this problem. On one hand, WhP was solved for surface groups in [8] and, recently, for hyperbolic groups in [6]. On the other hand, the generalized WhP has been also solved for torsion free hyperbolic groups (and so, finitely generated free groups) in [1].

Let $\mathcal{G}$ denote the class of toral relatively hyperbolic groups (i.e. torsion-free relatively hyperbolic groups with abelian parabolic subgroups). The main result in this note is the following.

Theorem 1.1. *The WhP is solvable for $G \in \mathcal{G}$.*

This result implies that the WhP is solvable for limit groups and torsion-free hyperbolic groups. Notice, that in [6] the WhP was solved as a consequence of the solution of the isomorphism problem for hyperbolic groups. For every two tuples of elements $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ of G one has to construct JSJ decompositions of auxiliary groups $G_1(u_1, \dots, u_n)$ and $G_2(v_1, \dots, v_n)$ obtained from G and then decide whether G_1 and G_2 are isomorphic. The algorithm in [6] for the construction of the JSJ decomposition involves complete enumeration of all presentations of the group obtained by Tietze transformations. The advantage of our approach is that we have to construct the JSJ decomposition only for the original G and only once, therefore it has lower complexity.

Key words and phrases. Toral relatively hyperbolic groups; Whitehead problem, automorphism.

We begin the proof by mentioning that in 1984, Collins and Zieschang extended Whitehead's methods to free products of finitely many freely indecomposable groups, assuming that WhP can be solved in each factor [2],[3]. Therefore, without loss of generality, we can restrict ourselves to consider only the case when G is freely indecomposable. Groups from this class have algorithmically computable canonical $Out(G)$â-invariant abelian JSJ decompositions with all parabolic subgroups being elliptic, see [4]. Hence, we can assume that G comes with such a JSJ decomposition explicitely given to us.

2. The algorithm

Definition 2.1. Let $G = A *_C B$ be an elementary abelian splitting of a freely indecomposable group G. For $c \in C$ we define an automorphism $\phi_c : G \to G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(b) = b^c = c^{-1}bc$ for $b \in B$.

If $G = A*_C = \langle A, t | c^t = c', c \in C\rangle$ (where c and c' represent the images of the same element of C under the two given inclusions $\alpha, \omega \colon C \to A$) then for $c \in C$ define $\phi_c : G \to G$ such that $\phi_c(a) = a$ for $a \in A$ and $\phi_c(t) = ct$.

In both cases, we call ϕ_c a *Dehn twist* obtained from the corresponding elementary abelian splitting of G.

Note that, if $G = A *_C B$, then every automorphism of B acting trivially on C can be extended to a unique automorphism of G acting trivially on A.

Definition 2.2. Let G be a freely indecomposable group, and let $\Gamma(V, E; T)$ be an Abelian JSJ decomposition of G (computable from a given presentation for G). We define the group $Out_\Gamma(G)$, to be the subgroup of $Out(G)$ generated by the following types of automorphisms of G:

1. Dehn twists along edges in Γ,
2. automorphisms of an abelian vertex group that preserve the peripheral subgroups of the group,
3. automorphisms of a QH-vertex group G_u preserving the peripheral subgroups of the group, up to conjugacy (geometrically, these are Dehn twists along simple closed curves on the punctured surface Σ with $\pi_1(\Sigma) \cong G_u$).

The full preimage of $Out_\Gamma(G) \leqslant Out(G)$ in $Aut(G)$ (which, of course, contains all inner automorphisms) is called the *group of canonical automorphisms with respect to* Γ, denoted $AutC_\Gamma(G)$.

Lemma 2.3. [11] *With the notation of Definition 2.2, $[Out(G) : Out_\Gamma(G)] < \infty$ and hence, the group of canonical automorphisms of G has finite index in the group of all automorphisms of G, $[Aut(G) : AutC_\Gamma(G)] < \infty$.*

The following proposition implies that one can effectively find representatives of all conjugacy classes of automorphisms of rigid subgroups compatible with edge groups.

Proposition 2.4. *[Theorem 5.11, [4]] Let G (respectively, H) be a toral relatively hyperbolic group, and let $\mathcal{A} = (A_1, \dots, A_n)$ (resp., $\mathcal{B} = (B_1, \dots, B_n)$) be a finite list of non-conjugated maximal abelian subgroups of G (resp., H) such that the abelian decomposition of G modulo $\mathcal{A}$ (resp. of H modulo $\mathcal{B}$) is trivial. The number of conjugacy classes of monomorphisms from G to H that map subgroups from $\mathcal{A}$ onto conjugates of the corresponding subgroups from $\mathcal{B}$ is finite. A set of representatives of the equivalence classes can be effectively found.*

If $G = H$, then there is at most a finite number of conjugacy classes of automorphisms compatible with the peripheral structure, and there is an algorithm to find representatives of all of them.

We can suppose that G in not abelian and not a closed surface group. We compute a canonical JSJ decomposition $\mathcal{D}$ for G, with the extra property that parabolic subgroups are elliptic. Notice that $AutC_{\mathcal{D}}(G)$ consists of automorphisms ϕ that map every vertex group of $\mathcal{D}$ into a conjugate of itself and have the following property: for any rigid subgroup H there exists $g \in G$ such that $\phi(h) = ghg^{-1}$ for any $h \in H$. By Lemma 2.3, $[Aut(G) : AutC_{\mathcal{D}}(G)] < \infty$. Every automorphism of G maps H to a conjugate of a rigid subgroup, and there is only a finite number (up to conjugation) of automorphisms of a rigid subgroup onto itself preserving its peripheral subgroups up to conjugacy. We can effectively find all such automorphisms and, therefore, compute left coset representatives $\tau_1, \dots, \tau_k$ of $AutC_{\mathcal{D}}(G)$ in $Aut(G)$. Then, to decide whether the tuple $(u_1, \dots, u_n)$ is in the orbit of the tuple $(v_1, \dots, v_n)$ with respect to $Aut(G)$, we have to decide whether $(\tau_i(u_1), \dots, \tau_i(u_n))$ is in the orbit of $(v_1, \dots, v_n)$ with respect to $AutC_{\mathcal{D}}(G)$, for some τ_i. Therefore, to solve the WhP in G we are reduced to solving the WhP for the group of canonical automorphisms $AutC(G)$.

Combining foldings and slidings, we can transform the JSJ decomposition $\mathcal{D}$ in such a way that each non-cyclic abelian vertex group that is connected to a rigid subgroup is connected to only one vertex group and this vertex group is rigid. We fix such a decomposition and denote it again by $\mathcal{D}$. We also fix a maximal forest T_1 joining all non-abelian vertex groups, and a maximal subtree T of $\mathcal{D}$ with $T_1 \subseteq T$. From now on, all canonical automorphisms will be with respect to $\mathcal{D}$. We order edges in T_1 and take free products with amalgamation following this order; then, we order the rest of the edges of $\mathcal{D}$ that are not in T, assign stable letters to these edges and take HNN extensions in this order. After that, we order edges in $T - T_1$.

Lemma 2.5. *Let $C =< c >, D =< d >$, $C \neq D$ be edge groups of a QH-subgroup Q. For any $u, v \in Q$ there exists a bound on possible numbers m, n for which there exists an automorphism α of Q with $\alpha(u) = d^{m\delta} v c^{n\gamma}, \alpha(c) = c^{\gamma}, \alpha(d) = d^{\delta}$, for some $\gamma, \delta \in Q$. Moreover, there exists an algorithm to find such a bound, all valid values of m, n and, for each pair m, n, an automorphism α.*

Proof. The proof is similar to the proof of Lemmas 3.4 and 3.5 in [8]. Notice that, under the assumptions $\alpha(d) = d^{\delta}$ and $\alpha(c) = c^{\gamma}$, $\alpha(u) = d^{m\delta} v c^{n\gamma}$ iff $\alpha(d^{-m} u c^{-n}) = v$. We choose a base point P on the boundary corresponding to C and represent u, v as closed curves on the surface. Moreover, we take minimal representatives in the sense of [8]. Then minimal representatives for v and $d^{-m} u c^{-n}$ must have the same number of self-intersection points. The existence of such α can be effectively verified as in [8]. □

Remark 2.6. For each $u, v, \gamma \in Q$ there is at most one number n for which there exists α with the properties that $\alpha(u) = vc^{n\gamma}$ and $\alpha(c) = c^{\gamma}$. Indeed if, in addition, $\beta(u) = vc^{m\gamma}$ and $\beta(c) = c^{\gamma}$ for some others m and β, then $\beta\alpha^{-1}(vc^{n\gamma}) = vc^{m\gamma}$ and $\beta\alpha^{-1}(c^{\gamma}) = c^{\gamma}$. Now, choosing the basepoint P on c^{γ}, the curves $vc^{n\gamma}$ and $vc^{m\gamma}$ have different number of self-intersections unless $n = m$.

A multiple version of Lemma 2.5 gives the following lemma.

Lemma 2.7. *Let $C=<c>, D=<d>$, $C\neq D$ be edge groups of a QH-subgroup Q. For any finite set I and tuples of elements $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ from Q, there exists a bound on possible numbers m_i, n_i for which there exists an automorphism α of Q with $\alpha(u_i)=d^{m_i\delta}v_ic^{n_i\gamma}, \alpha(c)=c^\gamma, \alpha(d)=d^\delta$, for some $\gamma,\delta\in Q$. Moreover, there exists an algorithm to compute such bound, all valid values of m_i, n_i and, for each pair of tuples $(m_i)_{i\in I}$ and $(n_i)_{i\in I}$, an automorphism α.*

Lemma 2.8. *Let $G\in\mathcal{G}$, and take elements $v,w\in G$ and an abelian subgroup $C\leqslant G$. If either v or w do not belong to the maximal abelian subgroup containing C, then there exists at most one pair of elements $\gamma_1,\gamma_2\in C$ such that $w=\gamma_1v\gamma_2$; furthermore, there is an algorithm deciding whether it exists or not and, in the affirmative case, computing such elements $\gamma_1,\gamma_2\in C$.*

Proof. Assume v (or w) does not belong to the maximal abelian subgroup C' of G containing C, and suppose $w=\gamma_1v\gamma_2=\gamma_3v\gamma_4$, for some $\gamma_1,\gamma_2,\gamma_3,\gamma_4\in C$. Then, $v^{-1}(\gamma_3^{-1}\gamma_1)v=\gamma_4\gamma_2^{-1}$. By the CSA property of toral relatively hyperbolic groups (see Lemma 2.5 in [7]), C' is malnormal and so, $\gamma_1=\gamma_3$ and $\gamma_2=\gamma_4$.

To make the decision algorithmic, we remind that equations are solvable in toral relatively hyperbolic groups, see [5]. Therefore, we can decide whether such γ_1,γ_2 exist or not in G (the fact that $\gamma_i\in C$ can be expressed by the equation $[c,\gamma_i]=1$). □

Definition 2.9. Let $\mathcal{D}$ be an abelian JSJ decomposition of a freely indecomposable $G\in\mathcal{G}$ with a graph Γ that does not have abelian vertices. Let Γ_1 be a connected subgraph of Γ, and B be the fundamental group of Γ_1, $B\leqslant G$. An automorphism of B is called $\mathcal{D}$-compatible if it takes vertex subgroups of Γ_1 into conjugates of themselves, and edge subgroups of these vertices into conjugates of edge subgroups. Let $C=G_e$ be an edge group in $\mathcal{D}$, $e\notin\Gamma_1$, $K=G_{e'}$ be different edge group, $e'\notin\Gamma_1$, and suppose that for any $u,v\in B$ there exists only finitely many elements $c\in C$ and $k\in K$ such that u is taken to $k^\delta vc^\gamma$ by a $\mathcal{D}$-compatible automorphism of B sending c to c^γ and k to k^δ. We say that the *special Whitehead problem* with respect to K,C is solvable if K,C satisfy this property and for any $u,v\in B$ there is an algorithm to decide whether there exist $\gamma,\delta\in B, k\in K, c\in C$ such that u is taken to $k^\delta vc^\gamma$ by a $\mathcal{D}$-compatible automorphism of B sending c to c^γ and k to k^δ and to find all such γ,δ,k,c and the corresponding automorphism. If, instead of u,v, the same is true for tuples of elements $(u_1,\ldots,u_m)$ and $(v_1,\ldots,v_m)$, we say that the special Whitehead problem with respect to K,C (SWhP(K,C)) is solvable for tuples.

Lemma 2.10. *Let $\mathcal{D}$ be an abelian JSJ decomposition of a freely indecomposable group G without abelian vertex groups. Suppose B (as in Definition 2.9) has solvable $SWhP(K,C)$ for tuples, where $C=G_e, K=G_{e'}$ are edge groups of Γ, and D is the vertex group in the abelian decomposition $\mathcal{D}$ corresponding to the other endpoint of e not in Γ_1 (D is then either a MQH subgroup Q or a rigid subgroup R). Then $B*_CD$ has solvable $SWhP(K,C_1)$ for tuples, for any edge group $C_1=G_{e_1}$ $(e\neq e_1)$ of $\mathcal{D}$ belonging to D.*

Proof. It is enough to prove the lemma for the subgroup of canonical automorphisms fixing D. Denote it by $AutC_D(G)$. Let $\alpha\in AutC_D(G)$. The restrictions of α to all QH-subgroups are automorphisms that map edge subgroups into their conjugates. If $D=R$ is a rigid subgroup, then the statement follows from Lemma 4 because α acts trivially on D.

If D is a QH subgroup, then we can assume that α maps it to itself, and maps C to itself element-wise.

Since α is not a conjugation on B, $e' \neq e$. Suppose, first that $u, v \in D$. Let $c_1 \in C_1$, suppose that α is a $\mathcal{D}$-compatible automorphism of D such that $\alpha(u) = v{c_1}^{n\gamma}$, $\alpha(c_1) = {c_1}^{\gamma}$. There is only a finite number of possible such n. It follows from Lemma 3.5 [8] that $c_1^{n\gamma}$ can be effectively found. By Lemma 3.4 [8], SWhP(C_1) is solvable in D for tuples. For any two tuples $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$, there are finitely many combinations ${c_1}^{n_1}, \ldots, {c_m}^{n_m} \in C_1$ such that $u_1, \ldots, u_m$ can be taken to $v_1 c^{n_1\gamma}, \ldots, v_m c^{n_m\gamma}$.

Let now, $u = b_1 d_1 \cdots b_n d_n$ and $v = \bar{b}_1 \bar{d}_1 \cdots \bar{b}_n \bar{d}_n$, where $b_i, \bar{b}_i \in B$, $d_i, \bar{d}_i \in D$ be normal forms of u and v in $B *_C D$.

We assume, first that C and C_1 are not conjugate in D. Without loss of generality, we can assume that u and v are cyclically reduced. Every $\mathcal{D}$-compatible automorphism α of $B *_C D$ taking u to $k^{\delta} v \gamma^{\sigma}$, $k \in K$, should act as follows:

$$
\begin{aligned}
&\alpha(b_1) = k^{\delta} \bar{b}_1 c^{k_1}, \\
&\alpha(d_i) = c^{-k_i} \bar{d}_i c^{m_i} && \text{for } i = 1, \ldots, n-1, \\
&\alpha(b_i) = c^{-m_{i-1}} \bar{b}_i c^{k_i} && \text{for } i = 2, \ldots, n, \\
&\alpha(d_n) = c^{-k_n} \bar{d}_n \gamma^{\sigma}.
\end{aligned}
$$

Moreover, the number of possible values for $\gamma, k,\ k_1, k_n$ is finite. Therefore the number of possible values for k_i, m_i is finite by Lemma 2.5 [8]. Since $SWhP(K, C)$ is solvable for tuples in B and $SWhP(C, C_1)$ is solvable for tuples in D, one can decide whether some $\mathcal{D}$-compatible automorphisms α of B and β of D and a tuple of integers $(k_1, m_1, \ldots, k_n, m_n)$ exist such that

$$
\begin{aligned}
&\alpha(b_1) = k^{\delta} \bar{b}_1 c^{k_1}, \\
&\beta(d_i) = c^{-k_i} \bar{d}_i c^{m_i} && \text{for } i = 1, \ldots, n-1, \\
&\alpha(b_i) = c^{-m_{i-1}} \bar{b}_i c^{k_i} && \text{for } i = 2, \ldots, n, \\
&\beta(d_n) = c^{-k_n} \bar{d}_n \gamma^{\sigma}.
\end{aligned}
$$

If they exist then there also exists a $\mathcal{D}$-compatible automorphism of $B *_C D$ fixing C_1 and taking u to $k^{\delta} v \gamma^{\sigma}$, $k \in K, \gamma \in C_1$; otherwise, it does not exist.

Now we consider the case when C and C_1 are conjugate in D. In this case, we can assume $C = C_1$. Hence, every $\mathcal{D}$-compatible automorphism α of $B *_C D$ fixing C_1 and taking u to $k^{\delta} v \gamma^{\sigma}$, $\gamma \in C_1$ should act as follows:

$$
\begin{aligned}
&\alpha(b_1) = \gamma_0 k^{\delta} \bar{b}_1 c^{k_1}, \\
&\alpha(d_i) = c^{-k_i} \bar{d}_i c^{m_i} && \text{for } i = 1, \ldots, n-1, \\
&\alpha(b_i) = c^{-m_{i-1}} \bar{b}_i c^{k_i} && \text{for } i = 2, \ldots, n, \\
&\alpha(d_n) = c^{-k_n} \bar{d}_n \gamma^{\sigma},
\end{aligned}
$$

where $\gamma_0, \gamma, c \in C$, $i = 1, \ldots, n$.

Since we can post-compose the restriction of α on B and on D with conjugation by γ_0, the question of finding such α is equivalent to the problem of finding α when $\gamma_0 = 1$. And that problem have been considered in the previous case.

The lemma is proved for elements. The same proof works similarly if, instead of u and v, we consider tuples of elements. $\square$

Lemma 2.11. *Suppose B has solvable $SWhP(K_1, C_1)$ for tuples for any edge groups K_1, $C_1 = G_{e_1}$ of Γ_1, and D is a (non-abelian) vertex group in the abelian decomposition $\mathcal{D}$ not in Γ_1. Then $B *_C D$ has solvable $SWhP(K_2, C_2)$ for tuples, for any edge groups K_2, $C_2 = G_{e_2}$ of $\mathcal{D}$.*

Proof. If e_2 is an edge outgoing from the vertex with vertex group D, then the statement follows from the previous lemma. If e_2 is an edge outgoing from the vertex with vertex group in Γ_1, then C_2 is an edge group of Γ_1. Then we can use the fact that B has solvable $SWhP(C_1)$ for tuples for any edge group $C_1 = G_{e_1}$ of Γ_1, in particular for C_2, and write a proof similar to the proof of the previous lemma with $u = b_1 d_1 \cdots b_n$ and $v = \bar{b}_1 \bar{d}_1 \cdots \bar{b}_n$. □

Lemma 2.12. *Let B be the fundamental group of a connected subgraph Γ_1 of Γ. $SWhP(K, C)$ is solvable for tuples in B for any edge groups K, C of Γ.*

Proof. We use Lemma 2.11 and add to Γ_1 by induction edges that do not belong to the maximal subtree. Let $u_1, \dots, u_n, v_1, \dots, v_n \in B$, and compute normal forms for the conjugacy classes of $u_1, \dots, u_n$ and $v_1, \dots, v_n$ with respect to the last HNN-extension, $B = H_{*D} =< H, t | d^t = d', d \in D >$. Denote these normal forms by $\bar{u}_1, \dots, \bar{u}_n$ and $\bar{v}_1, \dots, \bar{v}_n$. Consider all simultaneous conjugates of normal forms of $u_1, \dots, u_n$, $i = 1, \dots, k$, that have the same syllable structure as $\bar{v}_1, \dots, \bar{v}_n$. We can do this because the membership problem in maximal abelian subgroups of B is solvable, therefore we can decide when two elements belong to the same coset of the edge group. If there is no such conjugate, then $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are not in the same orbit of $AutC(G)$. Otherwise, make a list of all of them and let us check, one by one, whether they are in the same $AutC_H(G)$-orbit as $\bar{v}_1, \dots, \bar{v}_n$, where $AutC_H(G)$ is the subgroup of the group of canonical automorphisms $AutC(G)$ fixing H. If one does then $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are in the same $AutC(G)$-orbit; otherwise, they don't. □

Proposition 2.13. *Let $\mathcal{D}$ be an abelian JSJ decomposition of a freely indecomposable $G \in \mathcal{G}$, with graph Γ, and let H be a designated vertex group in $\mathcal{D}$. Then the WhP is solvable for the group $AutC_H(G)$ of canonical automorphisms fixing H.*

Proof. We use induction on the number of abelian vertex groups, and the fact that no two abelian vertices are adjacent to each other (therefore we can transform the decomposition in such a way that every non-cyclic abelian subgroup is only connected to one non-abelian vertex group). The base of induction, namely the case when there is no abelian vertex groups follows from lemma 2.12.

Suppose we can solve the WhP for $AutC(P)$ when P has smaller number of abelian subgroups. We fix an abelian subgroup A. It is connected only to non-abelian vertex groups in $\mathcal{D}$, and let us distinguish two cases:

Case 1: A is connected to only one non-abelian vertex group in $\mathcal{D}$. Let $u_1, \dots, u_n$, $v_1, \dots, v_n \in G$. We compute normal forms of $v_1, \dots, v_n$ (with respect to the amalgamated product $P *_C A$). Denote them by $\bar{v}_1, \dots, \bar{v}_n$. Consider all simultaneous conjugates of normal forms of $u_1, \dots, u_n$ that have the same syllable structure as $\bar{v}_1, \dots, \bar{v}_n$. If there is no such conjugate, then $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are not in the same orbit of $AutC(G)$. Otherwise, make a list of all of them and let us check, one by one, whether they are in the same $AutC_P(G)$-orbit as $(\bar{v}_1, \dots, \bar{v}_n)$, where $AutC_P(G)$ is the subgroup of the group of canonical automorphisms $AutC(G)$ fixing P. If one of the conjugates of $(u_1, \dots, u_n)$ is in the same $AutC_P(G)$-orbit as $(\bar{v}_1, \dots, \bar{v}_n)$, then $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are in the same $AutC(G)$-orbit; otherwise, they are not.

To check whether the tuple $(u_1, \dots, u_n)$ is in the same $AutC_P(G)$-orbit as a given tuple $(v_1, \dots, v_n)$ with the same syllable structure, we represent for each i

the elements u_i, v_i in normal form as

$$u_i = a_0 r_1 a_1 r_2 \cdots a_{n-1} r_n a_n, \quad v_i = \bar{a}_0 \bar{r}_1 \bar{a}_1 \bar{r}_2 \cdots \bar{a}_{n-1} \bar{r}_n \bar{a}_n.$$

By induction, we can check whether there exists an automorphism sending $r_j, j = 1, \ldots, n$, to elements of the form $c_{1j} \bar{r}_j c_{2j}$, where $c_{1j}, c_{2j} \in C$. If it does not exist, then there is no automorphism sending u_i to v_i. If it exists then by Lemma 2.12 and Lemma 2.5 from [8] there is only a finite number of possible images for the r_i's. After finding and applying such an automorphism α, and assume that

$$u_i = a_0 r_1 a_1 r_2 \cdots a_{n-1} r_n a_n, \quad v_i = \hat{a}_0 r_1 \hat{a}_1 r_2 \cdots \hat{a}_{n-1} r_n \hat{a}_n.$$

It only remains to check whether we can extend α in such a way that $\hat{a}_0 = \alpha(a_0), \hat{a}_i = \alpha(a_i), \hat{a}_n = \alpha(a_n)$. If such extension doesn't exist, then u_i and v_i are not in the same orbit. The argument with tuples is similar.

Case 2: A is connected to several QH-subgroups. We represent A as $A = A_1 \times A_2$, where the subgroup generated in A by the edge groups has finite index in A_1. Canonical automorphisms map A_1 identically to itself modulo conjugation. We first add A_1 to P, denote the fundamental group of the obtained graph of groups by P_1, and prove that for any edge group K of Γ the problem $SWhP_1(K, A_1)$ is solvable for tuples. Then we consider $P_1 *_{A_1} A$ and repeat the argument done in the first case.

The proposition is proved. This also completes the proof of the theorem. □

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